Journal of Engineering Mathematics **33:** 201–217, 1998.

An ordinary differential equation for the Green function of time-domain free-surface hydrodynamics

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Received 22 April 1997; accepted in revised form 11 November 1997

Abstract. In this paper a general fourth-order ordinary differential equation is derived for a class of functions including the time-domain Green function of linearized free-surface hydrodynamics and all its spatial derivatives. Among all the applications following from this new result, the acceleration of numerical computations in BEM solutions of time-domain hydrodynamics was the initial motivation of this work. Two new alternative methods for the computation of convolution integrals based on the new ODEs are suggested and illustrated by a numerical example.

Key words: hydrodynamics, free surface, time domain, water waves, Green function, differential equation.

1. Introduction

When implementing numerical methods for solving linear water-wave problems in the time domain, one is most often led to compute convolution integrals involving the specific Green function of the problem. This function is the solution of Laplace's equation in the lower half space, together with the usual linearized Fourier–Robin condition on the undisturbed free-surface plane, and is originally given by an integral with an oscillating kernel over an infinite range (see 2.7). Its evaluation therefore requires heavy numerical computations and, in time-domain seakeeping codes, the major part of c.p.u. time is spent on these computations [1, 2].

The first numerical solutions of the water wave radiation problem in the time domain by boundary element methods (BEM) involving Kelvin singularities appeared in the early eighties in two dimensions [3, 4], and shortly after in 3D [5, 6, 7, 8, 9, 10]. In this period, several analytical studies were devoted to the elaboration of alternative formulations of the Green function that were more suitable for numerical calculation than the original integral definition. Routines based on these series and asymptotic expansions [11, 12, 13] are still commonly used for these calculations. A complementary trick was introduced later [14, 15] to decrease further the computation time of the function. Taking advantage of the fact that the integral to compute is a function of only two parameters varying in bounded domains (see 2.9), one evaluated the Green function by a bivariate interpolation in a table which was (pre)computed once for all and stored in a permanent file. This technique is naturally less accurate than the previous series expansions, but it allows a substantial cut in the overall computation time.

Our approach to the problem is rather different and, we believe, innovative. The velocity potential generated at a field point of the fluid domain by a source of time-varying strength is given by the convolution product of this strength and the Green function. It can therefore be considered as the output of a linear process, the input of which is the source intensity, while the Green function is the impulse-response function. From this point of view, it is natural in system theory to seek a differential model of such a system (or 'process') in order to replace the computation of convolution integrals by a simple integration of differential equations. Seeking the coefficients of a given model of the process is usually referred to as: identification. Once it has been completed, if the model is sufficiently accurate and the order not too large, the computational burden may be appreciably reduced. This could be called: a *system approach* to the problem.

Our first attempts which involved constant-coefficient differential equations to identify the time domain Green function [16, 17] were rather disappointing in terms of system size. The best models we obtained by this approach featured minimal orders of thirty, or more, and were nevertheless too inaccurate to be used in the whole parameter range, especially when both source and field points are close to the free surface. The model structure was obviously inadequate for this function and, as a result, we switched to variable-coefficient differential equations. The first results, derived numerically [18], were excellent and suggested that a fourth-order equation should be sufficient to cover the whole geometrical parameter range. Such an equation was then derived analytically. It was first published in [19] together with a numerical example.

A lemma generalizing this equation to all the spatial derivatives of the function is established in Section 3 of the present paper. In Section 4, initial conditions are derived. The particular equation for the time-domain Green function is derived and commented on Section 5. Then, differential equations for the first spatial derivatives of the Green function are obtained in Section 6. In Section 7, two alternative numerical methods are proposed for the computation of convolution integrals involving the Green function, and an example to illustrate their potential efficiency in terms of computing time is given.

2. The transient-hydrodynamics Green function

The usual assumptions of linearized theory of free-surface potential flows are made: the fluid is inviscid, surface tension is neglected, the flow is irrotational, the pressure is assumed constant above the free surface. Let $Q(x, y, z)$ denote a field point and $Q'(x', y', z')$ a source point, both lying in the lower half-space $(z \leqslant 0, z' \leqslant 0)$. Let us consider the flow due to a source of impulsive unit strength located in Q'. Let us define the origin of the time axis as the instant of existence of the source.

We will define the time domain Green function of the free-surface hydrodynamics problem as the solution of the following initial-boundary-value problem

$$\Delta_{x,y,z} G(x', y', z', x, y, z; t) = \delta(|QQ'|)\delta(t - t'), \quad z \leqslant 0; \quad t \geqslant 0, \tag{2.1}$$

$$\frac{\partial^2 G}{\partial t^2}(x', y', z', x, y, 0; t) + \frac{\partial G}{\partial z}(x', y', z', x, y, 0; t) = 0, \quad t \geqslant 0. \tag{2.2}$$

Condition at infinity

$$|\nabla_{x,y,z} G(x', y', z', x, y, z; t)| \to 0, \quad \left\{ \begin{array}{c} [(x - x')^2 + (y - y')^2] \to \infty \\ z \to -\infty \end{array} \right\}, \quad \forall t \geqslant 0. \tag{2.3}$$

Initial conditions

$$G(x', y', z', x, y, z; 0) = 0, \quad \frac{\partial G}{\partial t}(x', y', z', x, y, z; 0) = 0, \quad z \leqslant 0. \tag{2.4}$$

Equation (2.2) is the free surface condition, linearized at first order with respect to the wave steepness which is assumed to vanish. All the variables have been non-dimensionalized by proper length and time scales.

The solution of this initial-boundary-value problem, initially derived by Haskind [20] and Brard [21], was further generalized by Finkelstein [22] who also treated the finite-water-depth case, and gave complete formulations of the 2D solutions (see also Wehausen and Laitone [23]). In the present 3D infinite-water-depth case, the solution of (2.1–4) reads

$$G(x',y',z',x,y,z;t) = -\frac{1}{4\pi}\{\delta(t)G_0(x',y',z',x,y,z) + H(t)F(x',y',z',x,y,z;t)\} \tag{2.5}$$

with

$$G_0(x',y',z',x,y,z) = \left(\frac{1}{R} - \frac{1}{R_1}\right) \tag{2.6}$$

and

$$F(r,Z;t) = 2\int_0^\infty J_0(Kr)\,\mathrm{e}^{KZ}\sqrt{K}\,\sin[\sqrt{K}t]\,\mathrm{d}K, \tag{2.7}$$

where δ is the Dirac impulse, H the Heaviside step function, and J_0 a Bessel function of the first kind of order 0. Due to the form of (2.5) with regard to the time variable, G_0 is often referred to as the instantaneous or impulsive part of the Green function, while F is called the memory part. In the present paper our attention will be focused on this later part and, for the sake of brevity, it will be mostly referred to as the *Green function* in the sequel. The new space variables in (2.6) and (2.7) are defined by

$$r = \sqrt{(x-x')^2 + (y-y')^2}; \quad R = \sqrt{r^2 + (z-z')^2};$$

$$Z = z + z'; \quad R_1 = \sqrt{r^2 + Z^2}.$$

By a simple change of variable $(KR_1 \to \lambda)$ in (2.7), Jami [5] showed that the memory part of the Green function can be expressed as a function of two real variables (μ, τ)

$$F(r,Z;t) = 2R_1^{-3/2}\tilde{F}(\mu,\tau) \tag{2.8}$$

with

$$\tilde{F}(\mu,\tau) = \int_0^\infty J_0(\lambda\sqrt{1-\mu^2})\,\mathrm{e}^{-\lambda\mu}\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,\tau)\,\mathrm{d}\lambda, \tag{2.9}$$

where $\mu = -Z/R_1$ and $\tau = t/\sqrt{R_1}$. Because μ is simply the cosine of the angle θ (see Figure 1), it will lie in the bounded domain: $0 \leqslant \mu \leqslant 1$, with $\mu = 0$ when (and only when) the two points Q and Q' lie on the free surface (*i.e.* $z = z' = 0$), and $\mu = 1$ when the two points belong to the same vertical axis (i.e. $r = 0$).

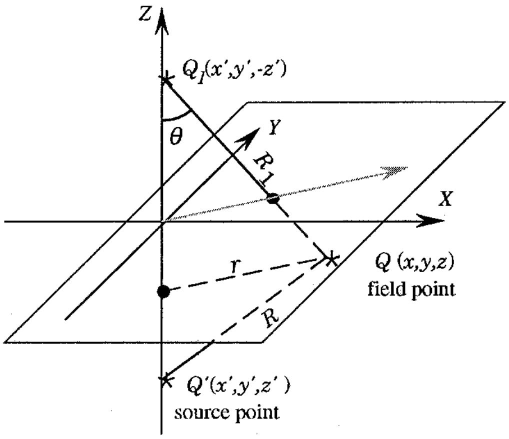

Figure 1. Definition sketch.

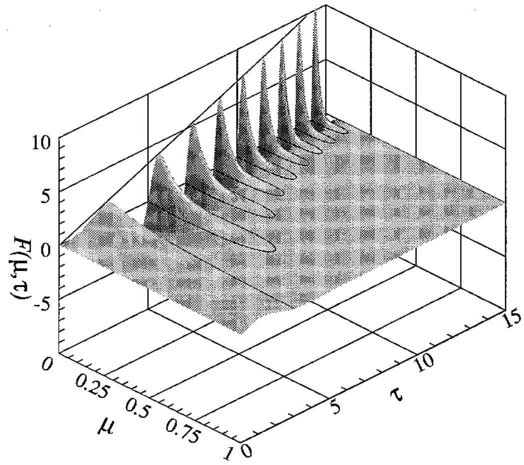

Figure 2. The Green function in its natural variables (μ, τ).

We shall denote μ and τ as the *natural* variables of the Green function, and $(r, Z; t)$ as the *initial* ones. The variable μ depends only on the relative position of the two points, while τ is the time-related variable. The ordinary differential equation we are seeking will be derived for $\tilde{F}$ with respect to the natural variable τ; then we shall return to the Green function F expressed in terms of its initial variables.

The function $\tilde{F}(\mu, \tau)$ is plotted on Figure 2 in the domain: $[0 \leqslant \mu \leqslant 1, 0 \leqslant \tau \leqslant 15]$. We notice the amplification of the oscillatory behaviour as μ approaches 0. In this limit, the function can be expressed as a combination of products of Bessel functions [23, pp. 608–609].

$$\tilde{F}(0, \tau) = \frac{\pi\tau}{2\sqrt{2}} \left[J_{1/4}\left(\frac{\tau^2}{8}\right) J_{-1/4}\left(\frac{\tau^2}{8}\right) + J_{3/4}\left(\frac{\tau^2}{8}\right) J_{-3/4}\left(\frac{\tau^2}{8}\right) \right] \frac{\tau^2}{8}, \tag{2.10}$$

from which the limit for $\tau \to \infty$ may be shown to oscillate between secular bounds

$$-\frac{\tau}{\sqrt{2}} \leqslant \tilde{F}(0, \tau \to \infty) \leqslant \frac{\tau}{\sqrt{2}}. \tag{2.11}$$

The upper bound is plotted as a black straight line in Figure 2. This singular behavour of the Green function is observed when μ is strictly zero, which occurs only when both source and field points are on the free surface (see Figure 1). Otherwise, considering for instance the series expansion of the Green function [13], we can see that the convergence for $\tau \to \infty$ is ensured for all $\mu > 0$ owing to a factor $\exp(-\mu\tau^2/4)$.

The locus of $\tilde{F}(\mu, \tau) = 0$ in the (μ, τ) plane is also plotted in Figure 2 in order to show that the number of zeros of $\tilde{F}$ remains finite for all strictly positive values of μ. The function is thus said to be non-oscillatory. The first time derivative, which represents the dynamic pressure in the fluid due to the impulsive source, has obviously the same behaviour, and the non-oscillatory decay of the free motion of a floating body in response to an initial velocity or displacement [24] probably results from this remarkable property.

At the other bound $\mu = 1$, the Bessel function disappears from the kernel of the integral in (2.9), and the Green function may be directly expressed as an Hermite polynomial, or by special functions like parabolic cylinder or confluent hypergeometric functions. Choosing this last form we get

$$\tilde{F}(1,\tau) = \tau \exp\left(-\frac{\tau^2}{4}\right) M\left(-\tfrac{1}{2}, \tfrac{3}{2}, \frac{\tau^2}{4}\right). \tag{2.12}$$

From this expression, we may derive a very simple second-order ordinary differential equation, using the general confluent equation [see (5.3)]. In the general case (*i.e*. when $0 \leqslant \mu < 1$), the differential equation will not be of second order, but of fourth order. It will be derived in Section 5 as a particular case of the lemma established in the next section.

3. A general differential equation

In this section, a general differential equation is derived from which particular equations will be obtained for both the Green function and its spatial derivatives.

LEMMA *Let v and l be two real parameters, τ and μ two real variables with* $0 < \mu \leqslant 1$. *The function $A_{v,l}(\mu,\tau)$ defined by*

$$A_{v,l}(\mu,\tau) = \int_0^\infty \lambda^l \,\mathrm{e}^{-\lambda\mu} J_v(\lambda\sqrt{1-\mu^2}) \sin(\sqrt{\lambda}\,\tau)\,\mathrm{d}\lambda \tag{3.1}$$

is a solution of the differential equation

$$\frac{\partial^4 A_{v,l}}{\partial\tau^4} + \mu\tau\frac{\partial^3 A_{v,l}}{\partial\tau^3} + \left(\frac{\tau^2}{4} + \mu(3+2l)\right)\frac{\partial^2 A_{v,l}}{\partial\tau^2}$$
$$+(l+\tfrac{5}{4})\tau\frac{\partial A_{v,l}}{\partial\tau} + ((l+1)^2 - v^2)A_{v,l} = 0. \tag{3.2}$$

Proof. Let us first express the second and fourth derivatives of $A_{v,l}(\mu,\tau)$ with respect to τ

$$\frac{\partial^2 A_{v,l}}{\partial\tau^2}(\mu,\tau) = -\int_0^\infty \lambda^{l+1}\,\mathrm{e}^{-\lambda\mu} J_v(\lambda\sqrt{1-\mu^2}) \sin(\sqrt{\lambda}\,\tau)\,\mathrm{d}\lambda,$$

$$\frac{\partial^4 A_{v,l}}{\partial\tau^4}(\mu,\tau) = \int_0^\infty \lambda^{l+2}\,\mathrm{e}^{-\lambda\mu} J_v(\lambda\sqrt{1-\mu^2}) \sin(\sqrt{\lambda}\,\tau)\,\mathrm{d}\lambda.$$

A first change of variable: $p = \lambda\tau^2$ is performed, yielding

$$A_{v,l}(\mu,\tau) = \frac{1}{\tau^{2(l+1)}}\int_0^\infty \mathrm{e}^{-p(\mu/\tau^2)} J_v\left(p\frac{\sqrt{1-\mu^2}}{\tau^2}\right) p^l \sin(\sqrt{p})\,\mathrm{d}p, \tag{3.3a}$$

$$\frac{\partial^2 A_{v,l}}{\partial\tau^2}(\mu,\tau) = -\frac{1}{\tau^{2(l+2)}}\int_0^\infty \mathrm{e}^{-p(\mu/\tau^2)} J_v\left(p\frac{\sqrt{1-\mu^2}}{\tau^2}\right) p^{l+1} \sin(\sqrt{p})\,\mathrm{d}p, \tag{3.3b}$$

$$\frac{\partial^4 A_{v,l}}{\partial\tau^4}(\mu,\tau) = \frac{1}{\tau^{2(l+3)}}\int_0^\infty \mathrm{e}^{-p(\mu/\tau^2)} J_v\left(p\frac{\sqrt{1-\mu^2}}{\tau^2}\right) p^{l+2} \sin(\sqrt{p})\,\mathrm{d}p. \tag{3.3c}$$

Introduction of the new variable $u = 1/\tau^2$ in (3.3a) leads to

$$A_{v,l}(\mu,\tau) = \int_0^\infty u^{l+1}\,\mathrm{e}^{-pu\mu} J_v(pu\sqrt{1-\mu^2}) p^l \sin(\sqrt{p})\,\mathrm{d}p. \tag{3.4}$$

Making use of the relation [25, 9.1.69 pp. 362], we can express the Bessel function in terms of the confluent hypergeometric function M (or *Kummer's* function, also referred to as Φ or ${}_1F_1$)

$$J_v(z) = \frac{(1/2z)^v\,\mathrm{e}^{-iz}}{\Gamma(v+1)} M(v+\tfrac{1}{2}, 2v+1, 2iz), \tag{3.5}$$

for $v \neq -1, -2$, where the Gamma function is singular. For these negative integer values of v, we may use the classical relation $J_{-n}(x) = (-1)^n J_n(x)$ [30 pp. 15] before continuing the derivation.

Introducing the notation

$$\alpha(\mu, v) = \frac{(1-\mu^2)^{v/2}}{2^v\Gamma(v+1)}, \tag{3.6}$$

for the coefficient function, we have now, from (3.4), (3.5) and (3.6)

$$A_{v,l} = \alpha(\mu,v)\int_0^\infty H_{v,l}(p,\mu,u) p^{l+v}\sin(\sqrt{p})\,\mathrm{d}p, \tag{3.7a}$$

$$-\tau^2\frac{\partial^2 A_{v,l}}{\partial\tau^2} = \alpha(\mu,v)\int_0^\infty H_{v,l}(p,\mu,u) p^{1+l+v}\sin(\sqrt{p})\,\mathrm{d}p, \tag{3.7b}$$

$$\tau^4\frac{\partial^4 A_{v,l}}{\partial\tau^4} = \alpha(\mu,v)\int_0^\infty H_{v,l}(p,\mu,u) p^{2+l+v}\sin(\sqrt{p})\,\mathrm{d}p, \tag{3.7c}$$

where

$$H_{v,l}(p,\mu,u) = u^{1+l+v}\,\mathrm{e}^{-up(\mu+i\sqrt{1-\mu^2})} M(v+\tfrac{1}{2}, 2v+1, 2iup\sqrt{1-\mu^2}). \tag{3.8}$$

Let us define some auxiliary functions and parameters

$$\begin{cases} A = -(1+l+v), \quad a = v+\tfrac{1}{2}, \quad b = 2v+1 = 2a, \\ f(u) = up(\mu+i\sqrt{1-\mu^2}), \quad \dot{f}(u) = p(\mu+i\sqrt{1-\mu^2}), \quad \ddot{f}(u) = 0, \\ h(u) = 2iup\sqrt{1-\mu^2}, \quad \dot{h}(u) = 2ip\sqrt{1-\mu^2}, \quad \ddot{h}(u) = 0. \end{cases} \tag{3.9}$$

Equation (3.8) may now be written in the simpler form

$$H_{v,l} = u^{-A}\,\mathrm{e}^{-f(u)} M(a,b,h(u)). \tag{3.10}$$

The function $H_{v,l}$ defined that way is known to satisfy the general confluent equation [25, 13.1.35 pp. 505] with respect to the variable u.

$$\begin{aligned} &\ddot{H} + \left[\frac{2A}{u} + 2\dot{f} + \left(b\frac{\dot{h}}{h} - \dot{h} - \frac{\ddot{h}}{\dot{h}}\right)\right]\dot{H} \\ &\quad + \left[\left(b\frac{\dot{h}}{h} - \dot{h} - \frac{\ddot{h}}{\dot{h}}\right)\left(\frac{A}{u} + \dot{f}\right) + \frac{A(A-1)}{u^2} + \frac{2A\dot{f}}{u} + \ddot{f} + \dot{f}^2 - \frac{a\dot{h}^2}{h}\right] H = 0. \end{aligned} \tag{3.11}$$

Substituting (3.9) in (3.11), we obtain

$$\frac{\partial^2 H_{v,l}}{\partial u^2} - \left(\frac{(1+2l)-2\mu u p}{u}\right)\frac{\partial H_{v,l}}{\partial u} + \left(p^2 - \frac{\mu p}{u}(1+2l) + \frac{l^2+(1+2l)-v^2}{u^2}\right)H_{v,l} = 0. \tag{3.12}$$

We can now return to the variable τ using

$$u = \frac{1}{\tau^2}, \quad \frac{\partial}{\partial u} = -\tfrac{1}{2}\tau^3\frac{\partial}{\partial \tau}, \quad \frac{\partial^2}{\partial u^2} = \tfrac{3}{4}\tau^5\frac{\partial}{\partial \tau} + \tfrac{1}{4}\tau^6\frac{\partial^2}{\partial \tau^2}. \tag{3.13}$$

In this variable, the differential Equation (3.12) becomes

$$\tfrac{1}{4}\tau^6\frac{\partial^2 H_{v,l}}{\partial \tau^2} + [(l+\tfrac{5}{4})\tau^5 - \mu p\tau^3]\frac{\partial H_{v,l}}{\partial \tau} + [p^2 - \mu p(1+2l)\tau^2 + ((l+1)^2 - v^2)\tau^4]H_{v,l} = 0. \tag{3.14}$$

Moving the terms involving p to the right-hand side, we get

$$\tfrac{1}{4}\tau^6\frac{\partial^2 H_{v,l}}{\partial \tau^2} + (l+\tfrac{5}{4})\tau^5\frac{\partial H_{v,l}}{\partial \tau} + ((l+1)^2 - v^2)\tau^4 H_{v,l} = \mu p\tau^3\frac{\partial H_{v,l}}{\partial \tau} + \mu p(1+2l)\tau^2 H_{v,l} - p^2 H_{v,l}.$$

This differential equation is valid for all μ and p, independent of τ. Thus, it still holds after we multiply each side by $\alpha(\mu, v)p^{l+v}\sin\sqrt{p}$ and then integrate from 0 to infinity with respect to the variable p. Doing so, we note that the original function $A_{v,l}(\mu,\tau)$ appears directly in the left-hand side

$$\begin{aligned}
&\tfrac{1}{4}\tau^6\frac{\partial^2 A_{v,l}}{\partial \tau^2} + (l+\tfrac{5}{4})\tau^5\frac{\partial A_{v.l}}{\partial \tau} + ((l+1)^2 - v^2)\tau^4 A_{v,l} \\
&= \mu\tau^3\frac{\partial}{\partial \tau}\left[\alpha\int_0^\infty H_{v,l}\left(p,\mu,\frac{1}{\tau^2}\right)p^{1+l+v}\sin\sqrt{p}\,\mathrm{d}p\right] \\
&\quad + (1+2l)\mu\tau^2\alpha\int_0^\infty H_{v,l}\left(p,\mu,\frac{1}{\tau^2}\right)p^{1+l+v}\sin\sqrt{p}\,\mathrm{d}p \\
&\quad - \alpha\int_0^\infty H_{v,l}\left(p,\mu,\frac{1}{\tau^2}\right)p^{2+l+v}\sin\sqrt{p}\,\mathrm{d}p.
\end{aligned} \tag{3.15}$$

The right-hand-side integrals were defined in (3.7) as the second and fourth derivatives of $A_{v,l}(\mu,\tau)$; then we have

$$\begin{aligned}
&\tfrac{1}{4}\tau^6\frac{\partial^2 A_{v,l}}{\partial \tau^2} + (l+\tfrac{5}{4})\tau^5\frac{\partial A_{v,l}}{\partial \tau} + ((l+1)^2 - v^2)\tau^4 A_{v,l} \\
&= \mu\tau^3\frac{\partial}{\partial \tau}\left[-\tau^2\frac{\partial^2 A_{v,l}}{\partial \tau^2}\right] + (1+2l)\mu\tau^2\left[-\tau^2\frac{\partial^2 A_{v,l}}{\partial \tau^2}\right] - \left[\tau^4\frac{\partial^4 A_{v,l}}{\partial \tau^4}\right],
\end{aligned} \tag{3.16}$$

from which the final result is derived

$$A_{v,l}^{(4)} + \mu\tau A_{v,l}^{(3)} + (\tfrac{1}{4}\tau^2 + \mu(3+2l))A_{v,l}^{(2)} + (l+\tfrac{5}{4})\tau A_{v,l}^{(1)} + ((l+1)^2 - v^2)A_{v,l} = 0, \quad (3.17)$$

with the notation

$$A_{v,l}^{(n)} = \frac{\partial^n A_{v,l}}{\partial \tau^n}.$$

LIMIT CASE $\mu = 0$

The above derivation remains valid as long as the integral (3.1) and its time derivatives exist, which is ensured when μ is strictly positive. The existence of these integrals when $\mu = 0$ will be proved now, in the special cases $l = -\frac{1}{2} + p; v = q$, with p and q being two non-negative integers.

Let us define the auxiliary function $K_{p,q}(R,\tau)$ by

$$K_{p,q}(R,\tau) = \int_0^\infty \lambda^{-(1/2)+p} J_q(\lambda R) \sin(\sqrt{\lambda}\,\tau)\,\mathrm{d}\lambda, \quad (3.18)$$

where R is a positive real variable. The functions A and K are then linked by the relation

$$A_{v,l}(0,\tau) = K_{p,q}(1,\tau). \quad (3.19)$$

For $p = q = 0$, the integral $K_{0,0}(R,\tau)$ exists and is explicitly given in [26, pp. 609] by

$$K_{0,0}(R,\tau) = \frac{\pi^2}{\sqrt{R}} \frac{\tau}{2R} J_{1/4}\left(\frac{\tau^2}{8R}\right) J_{-(1/4)}\left(\frac{\tau^2}{8R}\right). \quad (3.20)$$

Differentiating (3.18) twice with respect to τ, we obtain

$$\frac{\partial^2 K_{p,q}}{\partial \tau^2}(R,\tau) = -K_{p+1,q}(R,\tau). \quad (3.21)$$

Thus, we can derive the integrals $A_{0,-(1/2)+p}(0,\tau)$ simply by applying (3.21) to (3.18) and expressing the result at $R = 1$.

For the second index q, the recursion is provided by the well-known formulas for the derivatives of the Bessel function [30, pp. 45] which give in our notations

$$\frac{q}{R} K_{p,q}(R,\tau) - \frac{\partial}{\partial R} K_{p,q}(R,\tau) = K_{p+1,q+1}(R,\tau). \quad (3.22)$$

Finally, all the integrals $A_{q,-(1/2)+p}(0,\tau)$ are finite and can be derived from (3.19) and (3.20) by simple recursive calculations. This allows us to extend the range of applicability of the lemma to the cases $\mu = 0$, when $l = -\frac{1}{2} + p; v = q$. Fortunately, the time-domain Green function which will be addressed in a later section, falls into this category.

4. Initial conditions

The integration of (3.17) requires the first four derivatives of $A_{v,l}(\mu,\tau)$ as initial conditions. These derivatives may be derived directly by substitution of $\tau = 0$ in the initial form (3.1) of the function, *i.e.*

$$A_{v,l}^{(2k)}(\mu,0) = 0, \quad A_{v,l}^{(2k+1)}(\mu,0) = (-1)^k \int_0^\infty \lambda^{l+(2k+1)/2}\, \mathrm{e}^{-\lambda\mu} J_v(\sqrt{1-\mu^2})\, \mathrm{d}\lambda; \quad k = 0, 1, \ldots. \tag{4.1}$$

All even derivatives, including the function itself, are zero at the origin. In the above expression of the odd derivatives, we can identify the integral formulation of the associated Legendre function of the first kind of degree β and order $-v$ which satisfy [26, 6.624.6 pp. 734]

$$\int_0^\infty \lambda^\beta\, \mathrm{e}^{-\lambda\mu} J_v(\lambda\sqrt{1-\mu^2})\, \mathrm{d}\lambda = \Gamma(\beta+v+1) P_\beta^{-v}(\mu) \tag{4.2}$$

provided

$$0 < \mu \leqslant 1; \quad \Re\mathrm{e}(\beta+v) > -1,$$

hence

$$A_{v,l}^{(2k)}(\mu,0) = 0, \quad A_{v,l}^{(2k+1)}(\mu,0) = (-1)^k \Gamma\left(l + \frac{2k+3}{2} + v\right) P_{l+k+1/2}^{-v}(\mu); \quad k = 0, 1, \ldots.$$

FUNCTION OF INTEGER ORDER AND INTEGER DEGREE

The applications of (3.17) and (4.4) we are presently dealing with involve only associated Legendre functions of integer order v and integer degree β. When v is a positive integer, we can return to an associated Legendre function of positive order, using [26, 8.752.2 pp. 1025]

$$\Gamma(\beta+v+1) P_\beta^{-v}(\mu) = (-1)^v \Gamma(\beta-v+1) P_\beta^v(\mu). \tag{4.3}$$

Hence, in that case, the general form of the derivatives of $A_{v,l}(\mu,\tau)$ at $\tau = 0$ is

$$A_{v,l}^{(2k)}(\mu,0) = 0, \quad A_{v,l}^{(2k+1)}(\mu,0) = (-1)^{k+v} \Gamma\left(l + \frac{2k+3}{2} - v\right) P_{l+k+1/2}^{v}(\mu); \quad k = 0, 1, \ldots, \tag{4.4}$$

with

$$0 < \mu \leqslant 1; \; \Re\mathrm{e}(l+k+v) > -\tfrac{3}{2}.$$

We can furthermore express these functions in terms of the vth derivatives of Legendre polynomials of degree β, $P_\beta(\mu)$, using [26, 8.752.1 pp. 1025]

$$P_\beta^v(\mu) = (-1)^v (1-\mu^2)^{v/2} \frac{\mathrm{d}^v}{\mathrm{d}\mu^v} P_\beta(\mu). \tag{4.5}$$

Also, the Γ function of integer argument becoming a factorial by

$$\Gamma\left(l+\frac{2k+3}{2}-v\right)=\left(l+\frac{2k+1}{2}-v\right)!$$

we finally get

$$\begin{aligned}&A_{v,l}^{(2k)}(\mu,0)=0,\\&A_{v,l}^{(2k+1)}(\mu,0)=(-1)^k\left(l+\frac{2k+1}{2}-v\right)!(1-\mu^2)^{v/2}\frac{\mathrm{d}^v}{\mathrm{d}\mu^v}P_{l+k+1/2}(\mu);\\&\quad k=0,1,\ldots\,.\end{aligned}\tag{4.6}$$

5. An ODE for the Green function

The Green function $\tilde{F}$ can now be expressed simply, from (3.1) and (2.9), as $A_{0,1/2}(\mu,\tau)$. Then, substituting these values $v=0, l=\frac{1}{2}$ of the parameters in (3.17) and (4.6), we get straightforwardly the following fourth-order differential equation for the Green function $\tilde{F}$ in the couple of natural variables (μ,τ).

$$\tilde{F}^{(4)}+\mu\tau\tilde{F}^{(3)}+(\tfrac{1}{4}\tau^2+4\mu)\tilde{F}^{(2)}+\tfrac{7}{4}\tau\tilde{F}^{(1)}+\tfrac{9}{4}\tilde{F}=0,\tag{5.1}$$

with the initial conditions

$$\tilde{F}^{(2k)}(\mu,0)=0,\quad \tilde{F}^{(2k+1)}(\mu,0)=(-1)^k(k+1)!P_{k+1}(\mu);\quad k=0,1,\ldots\,.\tag{5.2}$$

The coefficients in (5.2) which come from (4.6) could also have been derived directly from the expansion of the Green function into a series of Legendre polynomials given by Newman [11].

Having derived such a simple and compact equation we proceeded by checking its validity against some related analytical results and numerical algorithms. At the bound $\mu=0$ we could easily show that the analytical expression (2.10) of the Green function satisfies (5.1) by using the symbolic computation software MAPLE V.4.

At the opposite bound $\mu=1$, the function $\tilde{F}(1,\tau)$ satisfies a second-order differential equation, which can be easily derived from (2.12) and the general confluent Equation (3.11). Introducing a differential operator $\mathcal{L}$, we may express this ODE as

$$\mathcal{L}(\tilde{F}(1,\tau))=\left(2\frac{\partial^2}{\partial\tau^2}+\tau\frac{\partial}{\partial\tau}+3\right)(\tilde{F}(1,\tau))=0.\tag{5.3}$$

For this particular value of μ, it is noteworthy to observe that the fourth-order ODE (5.1) may be straightforwardly recovered by a simply ‘squaring’ of the operator $\mathcal{L}$

$$\mathcal{L}[\mathcal{L}(\tilde{F}(1,\tau))]=0=4\tilde{F}^{(4)}+4\tau\tilde{F}^{(3)}+(\tau^2+16)\tilde{F}^{(2)}+7\tau\tilde{F}^{(1)}+9\tilde{F}.\tag{5.4}$$

For intermediate values of the geometrical parameter μ in the range $[0,1]$, we did a numerical check by comparing the results obtained using either (5.1) or our standard Green-function routine based on series expansions. The first one being exact and the second an approximation, the excellent agreement we observed was simply a confirmation of the validity of the expansion, and of the accuracy of its computer implementation.

Returning to the initial variable set $(r, Z; t)$ from (5.1) through (2.8), we get the announced differential equation for F in a more useful form

$$(r^2 + Z^2)F^{(4)} - ZtF^{(3)} + (\tfrac{1}{4}t^2 - 4Z)F^{(2)} + \tfrac{7}{4}tF^{(1)} + \tfrac{9}{4}F = 0. \tag{5.5}$$

The derivation of this equation in Section 3, from the remarkable property of confluent hypergeometric functions to satisfy (3.11), was rather indirect. Once it was established, we returned to the initial-boundary-value problem (2.1)–(2.4) and sought a more direct derivation of the ODE from the initial equations of the problem. Unfortunately, so far these attempts have remained unsuccessful.

The Green function F may be regarded as the impulse response of a system with the source strength as an input and the velocity potential as an output. In literature of system theory, the derivatives of the impulse-response function at the origin of time are often referred to as the Markov parameters [27]. In the present case, these parameters of the Green-function process are all known from (5.2), up to an infinite order. This would give us potentially a deep insight into the dynamics of the system. At the moment, let us focus on the first four which provide us with the initial conditions necessary for the solution, or simulation of (5.5).

$$\begin{aligned} &F(r,Z;0) = 0, \quad \frac{\partial F}{\partial t}(r,Z;0) = 2\frac{\mu}{R_1^2} = -2\frac{Z}{(r^2+Z^2)^{3/2}}, \\ &\frac{\partial^2 F}{\partial t^2}(r,Z:0) = 0, \quad \frac{\partial^3 F}{\partial t^3}(r,Z;0) = 2\frac{-3\mu^2+1}{R_1^3} = 2\frac{r^2-2Z^2}{(r^2+Z^2)^{5/2}}. \end{aligned} \tag{5.6}$$

It may be helpful to write the fourth-order ODE (5.5) as a system of first-order equations. Keeping the superscript notation for the successive time derivatives, we shall write

$$\mathbf{Y}^{(1)} = [\mathbf{A}]\mathbf{Y}, \tag{5.7}$$

with $\mathbf{Y} = [F, F^{(1)}, F^{(2)}, F^{(3)}]^T$, and $[\mathbf{A}]$ the so-called companion matrix

$$[\mathbf{A}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(9/4R_1^2) & -(7t/4R_1^2) & (16Z-t^2)/4R_1^2 & Zt/R_1^2 \end{bmatrix}. \tag{5.8}$$

It is interesting to note that the determinant of $[\mathbf{A}]$ is time independent, and never vanishes, even when $\mu = 0$; it is thus always finite, except when Q and Q' coincide. Hence, the matrix will remain nonsingular and invertible, irrespective of the values of the variables.

Let $\{F_1(r,Z;t), F_2(r,Z;t), F_3(r,Z;t), F_4(r,Z;t)\}$ be the set of the four fundamental solutions of (5.5) satisfying unitary initial conditions $F_j^{(i)}(r,Z;0) = \delta_{i(j-1)}$ with $i = 0,\dots,3; j = 1,\dots,4$. Any solution of the ODE may be recovered as a linear combination of these basis functions which are plotted in Figure 3 in the natural variables (μ,τ). Being a solution of (5.5), the Green function can be expressed on this basis from (5.6) and we have

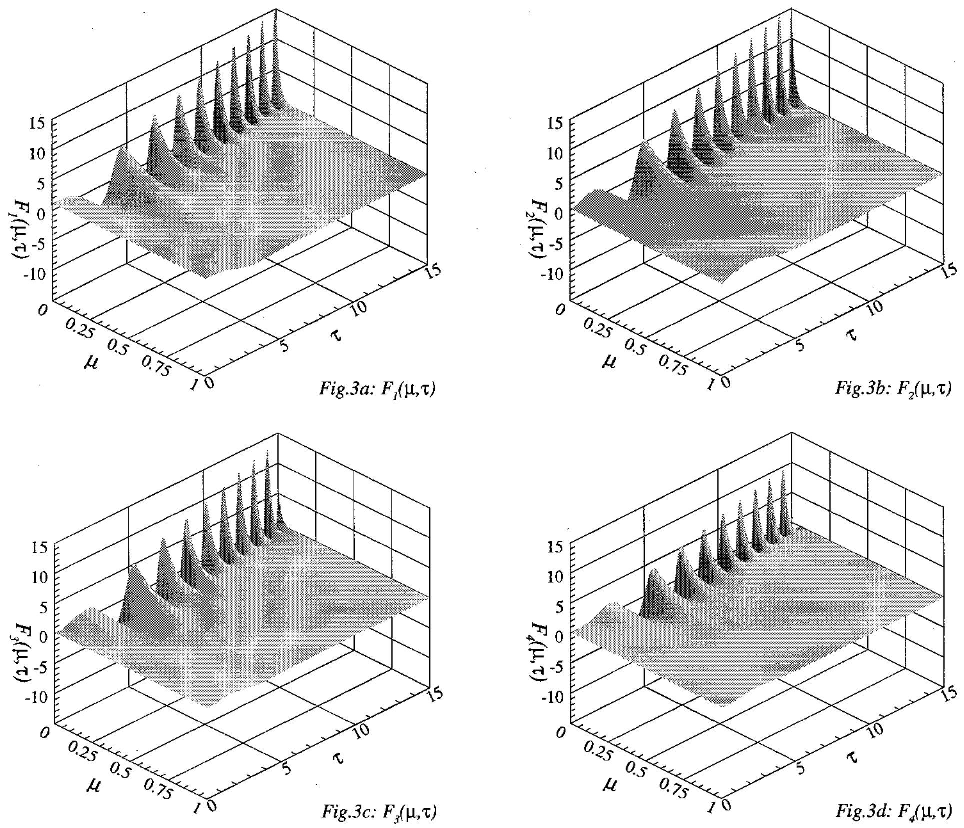

Figure 3. The fundamental solutions of the fourth order Green function ODE in their natural variables (μ, τ).

$$F(r, Z; t) = -2\frac{Z}{(r^2 + Z^2)^{3/2}} F_2(r, Z; t) + 2\frac{r^2 - 2Z^2}{(r^2 + Z^2)^{5/2}} F_4(r, Z; t). \tag{5.9}$$

The Wronskian $W(t)$ of the system (5.7) may be computed as the determinant of the state transition matrix [28]

$$W(t) = \det \begin{vmatrix} F_1 & F_2 & F_3 & F_4 \\ F_1^{(1)} & F_2^{(1)} & F_3^{(1)} & F_4^{(1)} \\ F_1^{(2)} & F_2^{(2)} & F_3^{(2)} & F_4^{(2)} \\ F_1^{(3)} & F_2^{(3)} & F_3^{(3)} & F_4^{(3)} \end{vmatrix}. \tag{5.10}$$

From the definition of the fundamental solutions we have $W(0) = 1$, and by Liouville's theorem [29]

$$W(t) = W(0) \exp\left\{ \int_0^t [Z\tau/R_1^2] \, \mathrm{d}\tau \right\} = \exp(Zt^2/2R_1^2). \tag{5.11}$$

Because Z is strictly negative when the source and the field points do not lie together on the free surface, the volume spanned by the solution of (5.7) in the four-dimensional space generated by the fundamental solutions will contract. In the limiting case $Z = 0$ it will not, as illustrated by the behaviour of the Green function for $\mu = 0$ (Figure 1).

6. ODEs for the gradient of the Green function

The solution of time-domain hydrodynamics problem by direct BEM method also involves the gradient of the Green function. Differential equations of the same kind are therefore needed for $\partial F/\partial r$ and $\partial F/\partial Z$ also.

Horizontal gradient

From (2.7) and the differentiation rule of the Bessel function we have

$$F_r(r, Z; t) = -2R_1^{-(5/2)} K(\mu, \tau), \tag{6.1}$$

with

$$K(\mu, \tau) = \int_0^\infty \lambda^{3/2}\, \mathrm{e}^{-\lambda\mu} J_1(\lambda\sqrt{1-\mu^2}) \sin(\sqrt{\lambda}\,\tau)\, \mathrm{d}\lambda = A_{1,3/2}(\mu, \tau). \tag{6.2}$$

Hence, the following fourth-order differential equation follows directly from the general lemma established in Section 3, with $v = 1$ and $l = \frac{3}{2}$

$$K^{(4)} + \mu\tau K^{(3)} + (\tfrac{1}{4}\tau^2 + 6\mu)K^{(2)} + \tfrac{11}{4}\tau K^{(1)} + \tfrac{21}{4}K = 0 \tag{6.3}$$

and from (4.6) the initial conditions

$$\begin{aligned} &K^{(2k)}(\mu, 0) = 0, \\ &K^{(2k+1)}(\mu, 0) = (-1)^k (k+1)! \sqrt{1-\mu^2} \frac{\mathrm{d}}{\mathrm{d}\mu} P_{k+2}(\mu), \quad k = 0, 1, \ldots . \end{aligned} \tag{6.4}$$

From these expressions and (6.1) we show that the horizontal gradient of the Green function satisfies, in the initial-variables set

$$(r^2 + Z^2)\frac{\partial^4 F_r}{\partial t^4} - Zt\frac{\partial^3 F_r}{\partial t^3} + (\tfrac{1}{4}t^2 - 6Z)\frac{\partial^2 F_r}{\partial t^2} + \tfrac{11}{4}t\frac{\partial F_r}{\partial t} + \tfrac{21}{4}F_r = 0, \tag{6.5}$$

with the initial conditions, up to the third order

$$\begin{aligned} &F_r(r, Z; 0) = 0, \quad \frac{\partial F_r}{\partial t}(r, Z; 0) = -6\mu\frac{\sqrt{1-\mu^2}}{R_1^3} = \frac{6rZ}{(r^2 + Z^2)^{5/2}}, \\ &\frac{\partial^2 F_r}{\partial t^2}(r, Z; 0) = 0, \quad \frac{\partial^3 F_r}{\partial t^3}(r, Z; 0) = (30\mu^2 - 6)\frac{\sqrt{1-\mu^2}}{R_1^4} = \frac{6r(4Z^2 - r^2)}{(r^2 + Z^2)^{7/2}}. \end{aligned} \tag{6.6}$$

Comparing (3.1) and (2.9), it is easy for us to realize that all higher derivatives of the Green function can be expressed as particular function $A_{v,l}(\mu, \tau)$, and that we could likewise derive a similar fourth-order differential equation for each of them by following the same steps.

Vertical gradient

The same derivation could be applied to the vertical derivative. Nevertheless, a shortcut is available, from (2.7), if we remark that

$$\frac{\partial F}{\partial Z} = -\frac{\partial^2 F}{\partial t^2}, \tag{6.7}$$

which is nothing but the free-surface condition extended to the whole lower half-space $(Z \leqslant 0)$. Hence, differentiating (5.5), twice, we directly obtain the result

$$(r^2 + Z^2)\frac{\partial^4 F_Z}{\partial t^4} - Zt\frac{\partial^3 F_Z}{\partial t^3} + (\tfrac{1}{4}t^2 - 6Z)\frac{\partial^2 F_Z}{\partial t^2} + \tfrac{11}{4}t\frac{\partial F_Z}{\partial t} + \tfrac{25}{4}F_Z = 0 \tag{6.8}$$

and, from (4.6), the initial conditions

$$\begin{aligned} &F_Z(r, Z; 0) = 0, \quad \frac{\partial F_Z}{\partial t}(r, Z; 0) = 2\frac{3\mu^2 - 1}{R_1^3} = \frac{4Z^2 - 2r^2}{(r^2 + Z^2)^{5/2}}, \\ &\frac{\partial^2 F_Z}{\partial t^2}(r, Z; 0) = 0, \quad \frac{\partial^3 F_Z}{\partial t^3}(r, Z; 0) = 2\frac{-15\mu^3 + 9\mu}{R_1^4} = \frac{-6Z(3r^2 - 2Z^2)}{(r^2 + Z^2)^{7/2}}. \end{aligned} \tag{6.9}$$

The similarity between the ODEs (6.5) and (6.8) for the two components of the gradient is remarkable. They differ only by the coefficient of the lowest order term which is $\frac{21}{4}$ for the former and $\frac{25}{4}$ for the latter.

7. Numerical applications of the Green-function ODE

The ODEs derived in the preceeding sections will be useful to accelerate the computation of convolution integrals involving the Green function or its derivatives; this was the primary motivation of the present study. In the BEM solution of time-domain hydrodynamics boundary-integral problems, the velocity potential (and/or its spatial gradient) induced at a field point Q by, say, a source of given strength $q(t)$ at Q' must be evaluated a very large number of times $(\sim O(10^8))$. Let us denote by $S(t)$ the memory part of this potential

$$S(t) = \int_0^t q(t')F(r, Z; (t - t'))\,\mathrm{d}t'. \tag{7.1}$$

As mentioned in Section 1, the computation of these convolution integrals in time-domain seakeeping codes constitutes the major part of the total numerical cost, due to the difficulties encountered in the evaluation of the kernel. Routines based on series expansions of (2.9) [11, 12, 13] are usually used; they are sometimes accelerated by the tabulation of the function $\tilde{F}(\mu, \tau)$ [14, 15]. The ability of these routines to deliver the value of the function whatever the time ordering in the calling sequence is not used in the applications considered herein. On the contrary, because these problems are solved by means of time-stepping procedures from initial conditions to the current time t, we always need to calculate the kernel sequentially with respect to the time variable, and never at random. This permits us to update the Green function in the integral kernel of convolution products by simply integrating the differential Equation (5.5) (or (6.3), (6.8), ..., for the derivatives), rather than computing it via series expansions or interpolation in the standard routines.

A few numerical tests of this new method have been performed on single integrals like (7.1) for different relative positions of Q and Q' and sinusoïdal input $q(t)$, and important reductions in computing time were observed. In these very first tests, however, the series-expansion routine was not fully optimized, and the ODE was integrated by a standard fourth-order Runge–Kutta procedure without any refinement or close checking of final accuracy. Then, we do not have currently enough elements to give a precise measure of the acceleration brought about by the present method to the sequential estimation of the Green function. A complete study of stability, accuracy versus time step and method order, $\ldots$, is in progress, and the results of these numerical investigations will appear in a forthcoming paper.

Another computational method based on the present differential equations was also considered. If the coefficients of the ODE were all constant with respect to time, a differential equation linking the output $S(t)$ to the input $q(t)$ of (7.1) could easily be derived from the knowledge of the differential Equation (5.5) for the impulse response, its derivatives at the origin (5.6), and the input initial conditions [see *e.g.* 27]. With such an equation, the output could be computed directly as a linear process simulation without any further computation of integrals such as (7.1).

Unfortunately, when the coefficients of the primary ODE are time varying like here, the relation between $q(t)$ and $S(t)$ is *a priori* an integro-differential equation featuring convolution integrals among the forcing terms. However, using expansions of the kernels, we can reduce it to a purely differential system, but of infinite dimension. If the dynamical information is concentrated in the lower-order terms, a finite-range sub-system can be defined by truncation, and used for the numerical simulation of the system as explained above. In those cases, since no further evaluations of the kernel are needed, the computation of the output may be considerably faster, depending obviously on the sub-system order. Futhermore, the differential equation being of short-memory form and the convolution integral of long-memory form, a large amount of computer-memory storage could also be saved.

A first attempt at using this second approach for the computation of (7.1) was presented in [19]. The proposed model had the form

$$\sum_{i=0}^{4}\left(\sum_{j=0}^{2} b_{ij}t^j\right) S^{(i)}(t) = \sum_{i=0}^{3}\left(\sum_{j=0}^{J} c_{ij}t^j\right) q^{(i)}(t). \tag{7.2}$$

The autoregressive terms on the left-hand side arose directly from (5.5). The right-hand-side order is imposed by the causality of the system [28, pp. 158]. The unknown coefficients of the forcing terms were obtained by subsitution of the successive time derivatives of (7.1) in (7.2). The resulting equation was then expressed at $t = 0$ in terms of the Markov parameters (5.2), (5.6). As explained above, this process leads to infinite-order polynomials, and must be terminated arbitrarily somewhere. With polynomials on the right-hand side of (7.2) truncated to degree one only, the numerical results were sometimes excellent and generally encouraging for a certain class of input (*i.e.* sinus function, high frequency). In Figure 4, the output of this t^1 model is compared to the output obtained by the classical approach (convolution integral by trapezoidal rule) for an input equal to $\sin(6t)$.

The same model, on the contrary, gave rather poor results, even divergent simulations, for low-frequency input. We derived higher-degree polynomials using a symbolic computation software; they did not noticeably improve the bandwidth of the method, and they seem less stable for long-term simulations. This inadequacy is probably caused by the fact that the

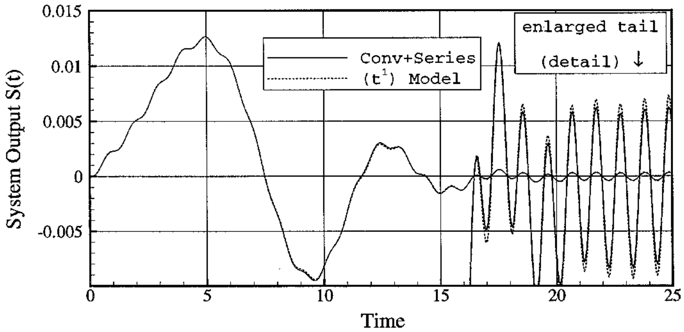

Figure 4. System output $S(t)$ computed by both methods. $M'(0,0,-1), M(5,0,-1)$ – [$\mu = 0{\cdot}3714, R_1 = 5{\cdot}3851, \delta t = 0{\cdot}05$].

method used to derive these low-degree models was purely algebraic and did not take the dynamics of the system into account.

We were nevertheless encouraged to pursue our investigations in that direction by results such as those plotted in Figure 4. So we will carry on with this study, but using now more elaborate models and identification methods.

8. Conclusion

A fourth-order ordinary differential equation and the corresponding initial conditions were derived for a class of functions including the Green function of time-domain hydrodynamics and all its spatial derivatives. The solution of linear problems for time-domain hydrodynamics requires the computation of convolution integrals involving this function. These computations represents the most c.p.u.-time-consuming part of the computer codes for solving linearized time-domain seakeeping problems by BEM. As a first application of our ODEs, an alternative method for the in-line evaluation of the Green function during the computation of the convolution integrals was proposed. Encouraging results have been obtained during preliminary numerical tests on single integrals. We are now implementing this method in an existing 3D BEM solver in order to measure the benefit we could expect in real engineering applications.

A further improvement based on a complete input-output differential model was also considered. Promising preliminary results were obtained for high-frequency input, but the bandwidth of the method, with the current form of the model, does not fit the whole useful range for the envisaged application. We are still developing this second alternative method for the computation of the convolution integrals related to time-domain hydrodynamics.

Furthermore, many theoretical and practical applications arising from the ODEs derived herein may be anticipated. We are particularly interested in their extension to the finite-water-depth problem for which only few formulations of the Green function, difficult to handle numerically, are available up to now. These theoretical developments have our continued attention.

Acknowledgements. This work was financially supported by the *Direction des Recherches, Études et Techniques* of the French Ministry of Defense (DGA).

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